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PHYSICOMATHEMATICAL MODELS FOR THEORIES OF NONDESTRUCTIVE TESTING OF THERMOPHYSICAL PROPERTIES

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The present work is devoted to the solution of some two-dimensional unsteady heat conduction problems for an infinite orthotropic cylinder with boundary conditions of the first- and second kind with a circular discontinuity line of temperature and specific heat flux. A solution of the two-dimensional unsteady heat conduction problem for an orthotropic cylinder is obtained with the aid of Laplace-Hankel transformations.

The recent years a great deal of attention has been paid to investigation of heat transfer in different bodies having local heat sources of different configurations on their surfaces [1-22]. Solutions of steady-state heat conduction problems of this type have been analyzed in detail in previous fundamental studies [2, 3]. The increasing interest in unsteady heat conduction problems for finite and semifinite bodies is due to the fact that solutions of these problems can find use in various theoretical and practical applications.

Of interest is the case of the distribution of the temperature fields in cylindrical axisymmetrical regions when the boundary conditions over the surface ($z = 0$) are determined as a Hankel integral for the Dirac delta function:

$$H_R(\delta_{r_0}) = \lim_{\varepsilon \to 0} \int\limits_{r_0}^{r_0+\varepsilon} r J_0(pr)\, q_\varepsilon(r)\, dr = \frac{J_0(pr_0)}{2\pi}, \tag{1}$$

where a unit density of heat flux is

$$q_\varepsilon(r) = \begin{cases} \dfrac{1}{\pi(2r_0+\varepsilon)\varepsilon}, & r_0 \le r \le r_0+\varepsilon, \\ 0, & r \notin [r_0, r_0+\varepsilon]. \end{cases}$$

Formula (1) will be adopted to prescribe the unit density of heat flux at some point over a body surface.

We now consider some physicomathematical models.

Problem 1. It is necessary to solve a differential heat conduction problem of the form

$$K_a \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T(r,z,\tau)}{\partial r}\right] + \frac{\partial^2 T(r,z,\tau)}{\partial z^2} = \frac{1}{a_z}\frac{\partial T(r,z,\tau)}{\partial \tau},$$

$$(0 \le r < R,\ 0 < z < h,\ \tau > 0) \tag{1.1}$$

under the following initial

$$T(r, z, 0) = T_0 = \text{const} \tag{1.2}$$

and boundary conditions (see Fig. 1)

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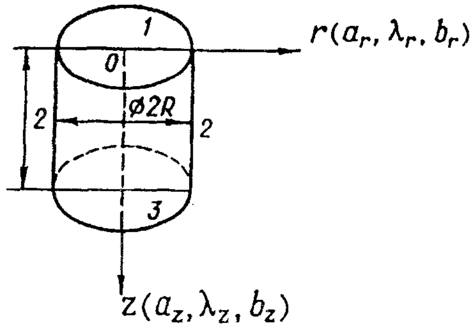

Fig. 1. Schematic representation of combinations of boundary conditions (1.3)-(1.5) over the surface of an infinite orthotropic cylinder: 1) $T(r, 0, \tau) = T_v(r, \tau)$, 2) $\partial T(R, z, \tau)/\partial r = -\alpha_R/\lambda_r[T(R, z, \tau) - T_0]$, 3) $\partial T(r, h, \tau)/\partial z = -\alpha_h/\lambda_z[T(r, h, \tau) - T_0]$.

$$T(r, 0, \tau) = T_v(r, \tau) \quad (0 \le r < R, \ z = 0, \ \tau > 0), \tag{1.3}$$

$$\left.\frac{\partial T}{\partial z}\right|_{z=h} = -\frac{\alpha_h}{\lambda_z}[T(r, h, \tau) - T_0] \quad (0 \le r < R, \ z = h, \ \tau > 0), \tag{1.4}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=R} = -\frac{\alpha_R}{\lambda_r}[T(R, z, \tau) - T_0] \quad (r = R, \ 0 < z < h, \ \tau > 0), \tag{1.5}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0 \quad (r = 0, \ 0 < z < h, \ \tau > 0). \tag{1.6}$$

A solution of the two-dimensional heat conduction problem (1.1)-(1.6) for a finite cylinder obtained with the aid of Laplace-Hankel transformations has the following form for the transform:

$$\overline{T}_{\mathrm{H}}(p, z, s) - \frac{T_0 R J_1(pR)}{ps} = \int_0^R r J_0(pr)\left[\overline{T}_v(r, s) - \frac{T_0}{s}\right] dr \times$$

$$\times \left\{\frac{\sqrt{K_a p^2 + s/a_z}\,\mathrm{ch}\left[\sqrt{K_a p^2 + s/a_z}\,(h - z)\right] + \alpha_h/\lambda_z\,\mathrm{sh}\left[\sqrt{K_a p^2 + s/a_z}\,(h - z)\right]}{\sqrt{K_a p^2 + s/a_z}\,\mathrm{ch}\left[\sqrt{K_a p^2 + s/a_z}\,h\right] + \alpha_h/\lambda_z\,\mathrm{sh}\left[\sqrt{K_a p^2 + s/a_z}\,h\right]}\right\}, \tag{1.7}$$

where

$$\overline{T}_{\mathrm{H}}(p, z, s) = \int_0^R r J_0(pr) \int_0^\infty T(r, z, \tau)\exp(-s\tau)\, d\tau dr;$$

$$\overline{T}_v(r, s) = \int_0^\infty T_v(r, \tau)\exp(-s\tau)\, d\tau.$$

The inverse Hankel transform for (1.7) is as follows

$$H^{-1}[\overline{T}_{\mathrm{H}}(p, z, s)] = \overline{T}(r, z, s) = \frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_0(p_n r)\,\overline{T}_{\mathrm{H}}(p_n, z, s)}{J_0^2(p_n R) + J_1^2(p_n R)},$$

where $\mu_n = p_n R$ are the roots of the characteristic equation

$$\mu_n J_1(\mu_n) - \mathrm{Bi_R} J_0(\mu_n) = 0$$

where $\mathrm{Bi}_R = \alpha_R R/\lambda_r$ is the Biot number.

When the thermophysical characteristics in the corresponding directions are equal, i.e., $\lambda_r = \lambda_z = \lambda$ and $a_r = a_z = a$ ($K_a = 1$), Eq. (1.7) represents a generalized solution of the two-dimensional unsteady heat conduction problem for an isotropic cylinder.

As an example of a particular application of the initial solution (1.7), we shall solve a two-dimensional unsteady heat conduction problem for a finite orthotropic cylinder under the following boundary conditions.

The initial temperature of the finite orthotropic cylinder is T_0. The initial temperature is assumed to be constant over the surface ($z = h$, $0 < r < R$ and $r = R$, $0 < z < h$), i.e., $T(r, h, \tau) = T(R, z, \tau) = T_0$. On the surface ($z = 0$, $0 < r < R$) the function $T(r, 0, \tau) = T_v(r, \tau)$ is prescribed.

It is necessary to determine the two-dimensional temperature field $T(r, z, \tau)$. Applying solution (1.8) for the case of α_h, $\alpha_R \to \infty$ (Bi_h, $\mathrm{Bi}_R \to \infty$), we arrive at

$$\overline{T}_{\mathrm{H}}(p, z, s) - \frac{T_0 R J_1(pR)}{ps} = \frac{\operatorname{sh}\left[\sqrt{K_a p^2 + s/a_z}\,(h - z)\right]}{\operatorname{sh}\left[\sqrt{K_a p^2 + s/a_z}\,h\right]} \int_0^R r J_0(pr)\left[\overline{T}_v(r, s) - \frac{T_0}{s}\right] dr. \tag{1.8}$$

In this case, the inverse Hankel transformation for solution (1.8) is as follows

$$H^{-1}\left[\overline{T}_{\mathrm{H}}(p, z, s)\right] = \overline{T}(r, z, s) = \frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_0(p_n r)}{J_1^2(p_n R)}\overline{T}_{\mathrm{H}}(p_n, z, s), \tag{1.8a}$$

where $p_n R = \mu_n$ are the roots of the equation $J_0(\mu_n) = 0$.

Using the inverse Laplace transform, solution (1.8) for the inverse transform $T(r, z, \tau)$ may be written as

$$T(r, z, \tau) - T_0 = \frac{a_z}{h}\sum_{n=1}^{\infty}\frac{J_0\left(\mu_n \frac{r}{R}\right)}{J_1^2(\mu_n)}\int_0^{\tau}\exp\left(-\mu_n^2\frac{a_r\xi}{R^2}\right)\times$$

$$\times\frac{\partial}{\partial z}\theta_0\left(\frac{h - z}{h}\left|\, i\pi\frac{a_z\xi}{h^2}\right.\right)\frac{2}{R^2}\int_0^R xJ_0\left(\mu_n\frac{x}{R}\right)\left[T_0 - T_v(x, \tau - \zeta)\right] dx d\xi, \tag{1.9}$$

where $\theta_0(\nu/\tau)$ is the theta-function [6, 22].

If the boundary conditions are set in discontinuity form

$$\overline{T}(r, z, s)|_{z=0} - \frac{T_0}{s} = \begin{cases}\overline{T}_v(r, s) - \dfrac{T_0}{s}, & r_1 > r > r_0;\\[2mm] 0, & r_1 < r < R \text{ and } 0 \le r < r_0,\end{cases} \tag{1.10}$$

solution (1.9) is as follows

$$T(r, z, \tau) - T_0 = \frac{a_z}{h}\sum_{n=1}^{\infty}\frac{J_0\left(\mu_n \frac{r}{R}\right)}{J_1^2(\mu_n)}\int_0^{\tau}\exp\left(-\mu_n^2\frac{a_r\xi}{R^2}\right)\times$$

$$\times\frac{\partial}{\partial z}\theta_0\left(\frac{h - z}{h}\left|\, i\pi\frac{a_z\xi}{h^2}\right.\right)\frac{2}{R^2}\int_{r_0}^{r_1} xJ_0\left(\mu_n\frac{x}{R}\right)\left[T_0 - T_v(x, \tau - \xi)\right] dx d\xi. \tag{1.11}$$

At $R \to \infty$ from (1.11) we obtain a solution for an infinite orthotropic plate under initial conditions (1.2):

$$\frac{\partial}{\partial z}\theta_0\left(\frac{h - z}{h}\left|\, i\pi\frac{a_z\xi}{h^2}\right.\right)\int_{r_0}^{r_1} x\exp\left(-\frac{x^2}{4a_r\xi}\right)I_0\left(\frac{rx}{2a_r\xi}\right)\left[T_0 - T_v(x, \tau - \zeta)\right] dx d\xi. \tag{1.12}$$

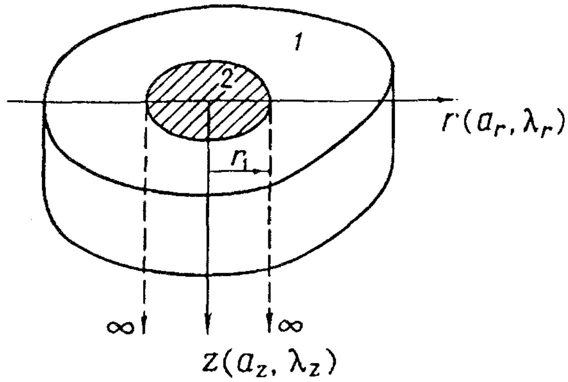

Fig. 2. Idealized physical model of semi-infinite body with a circular heat source on its surface ($z = 0$, $0 \le r < r_1$): 1) $T(r, 0, \tau) = T_0$, ($z = 0$, $r_1 < r < \infty$; 2) $T(r, 0, \tau) = T_v(r, \tau)$, ($z = 0$, $0 \le r < r_1$).

Solution (1.12) for the Laplace transform $\overline{T}_1(r, z, s)$ is written in the form (s is the Laplace transformation parameter):

$$\overline{T}_1(r, z, s) - \frac{T_0}{s} = \int_0^\infty pJ_0(pr) \frac{\operatorname{sh}\left[\frac{(h-z)}{\sqrt{a_z}}\sqrt{s + a_r p^2}\right]}{\operatorname{sh}\left[\frac{h}{\sqrt{a_z}}\sqrt{s + a_r p^2}\right]} \int_{r_0}^{r_1} xJ_0(px)\left[\overline{T}_v(x, s) - \frac{T_0}{s}\right] dxdp. \quad (1.13)$$

At $h \to \infty$ from (1.13) we obtain a two-dimensional unsteady solution $\overline{T}_2(r, z, s) = \lim_{h\to\infty} \overline{T}_1(r, z, s)$ for a semi-infinite orthotropic body under initial (1.2) and boundary (1.10) conditions

$$\overline{T}_2(r, z, s) - \frac{T_0}{s} = \int_0^\infty pJ_0(pr) \exp\left[-\frac{z}{\sqrt{a_z}}\sqrt{s + a_r p^2}\right] \int_{r_0}^{r_1} xJ_0(px)\left[\overline{T}_v(x, s) - \frac{T_0}{s}\right] dxdp. \quad (1.14)$$

Applying inverse Laplace transformation to (1.14), we arrive at

$$T_2(r, z, \tau) - T_0 = \frac{z}{4\sqrt{\pi}\, a_r \sqrt{a_z}} \int_0^\tau \frac{1}{\xi^{5/2}} \exp\left(-\frac{r^2}{4a_r\xi} - \frac{z^2}{4a_r\xi}\right) \times$$

$$\times \int_{r_0}^{r_1} x \exp\left(-\frac{x^2}{4a_r\xi}\right) I_0\left(\frac{rx}{2a_r\xi}\right) [T_v(x, \tau - \xi) - T_0]\, dxd\xi. \quad (1.15)$$

If the inner radius r_0 of the circular region tends to zero ($r_0 \to 0$), we obtain the following two-dimensional unsteady solution from (1.15) for the semiinfinite orthotropic body with initial (1.2) and boundary (1.10) conditions at $r_0 = 0$ (see Fig. 2):

$$T_2(r, z, \tau) - T_0 = \frac{z}{4\sqrt{\pi a_z}\, a_r} \int_0^\tau \frac{1}{\xi^{5/2}} \exp\left[-\left(\frac{r^2}{a_r} + \frac{z^2}{a_2}\right)\frac{1}{4\xi}\right] \times$$

$$\times \int_0^{r_1} x \exp\left(-\frac{x^2}{4a_r\xi}\right) I_0\left(\frac{rx}{2a_r\xi}\right) [T_v(x, \tau - \xi) - T_0]\, dxd\xi.$$

Assume that the excess temperature in the circular region ($0 \le r \le r_1$) over the body surface ($z = 0$) is constant, i.e., $T_v(r, \tau) - T_0 = T_c - T_0 = \text{const}$ ($T_0 \ne T_c$). Then it is easy to obtain an expression for the temperature difference $T_2(r, z, \tau) - T_0$ in the form

$$\frac{T_2(r, z, \tau) - T_0}{T_c - T_0} = \frac{1}{2}\int_0^{\infty} J_0\left(\frac{r}{r_1}x\right) J_1(x) \left\{ \exp\left(-\frac{z\sqrt{K_a}}{r_1}x\right) \operatorname{erfc}\left(\frac{z}{2\sqrt{a\tau}} - \frac{\sqrt{a_r\tau}}{r_1}x\right) + \right.$$

$$\left. + \exp\left(-\frac{z\sqrt{K_a}}{r_1}x\right) \operatorname{erfc}\left(\frac{z}{2\sqrt{a_z\tau}} + \frac{\sqrt{a_r\tau}}{r_1}x\right)\right\} dx .$$

At $r = 0$ we obtain a solution for $T_2(0, z, \tau)$ on the axis $z \geq 0$:

$$\frac{T_2(r, z, \tau) - T_c}{T_0 - T_c} = \operatorname{erf}\left(\frac{z}{2\sqrt{a_z\tau}}\right) + \frac{1}{\sqrt{\left(1 + \frac{r_1^2}{z^2K_a}\right)}} \operatorname{erfc}\left(\frac{r_1}{2\sqrt{a_r\tau}} + \sqrt{\left(1 + \frac{z^2K_a}{r_1^2}\right)}\right) .$$

The result of solving Problem 1 can be used for the development of nondestructive methods of determination of the thermophysical characteristics of various materials [1, 6, 24, 25].

Problem 2. It is necessary to solve differential heat conduction Eq. (1.1) for a finite cylinder under initial (1.2) and boundary (1.4)-(1.6) conditions. Instead of condition (1.3) on the surface $z = 0$ we have the following boundary condition

$$\left.\frac{\partial T(r, z, \tau)}{\partial z}\right|_{z=0} = \frac{\partial T(r, 0, \tau)}{\partial z} = -\frac{q(r, \tau)}{\lambda_z}, \tag{2.1}$$

($0 \leq r < R$, $z = 0$, $\tau > 0$), where $q(r, \tau)$ is the heat-flux density on the surface (see Fig. 1 on prescribing condition (2.1) instead of (1.3)).

A solution of Problem 2 for a finite orthotropic cylinder obtained with the aid of Laplace-Hankel transformations has the following form:

$$\bar{\theta}_H(p, z, s) = \bar{T}_H(p, z, s) - \frac{T_0 R J_1(pR)}{ps} = \int_0^R \frac{rJ_0(pr)\bar{q}(r, s)\,dr}{\lambda_z\sqrt{K_a p^2 + s/a_z}} \times$$

$$\times \left\{\frac{\sqrt{K_a p^2 + s/a_z}\,\mathrm{ch}\left[\sqrt{K_a p^2 + s/a_z}\,(h - z)\right] + \alpha_h/\lambda_z\,\mathrm{sh}\left[\sqrt{K_a p^2 + s/a_z}\,(h - z)\right]}{\sqrt{K_a p^2 + s/a_z}\,\mathrm{sh}\left[\sqrt{K_a p^2 + s/a_z}\,h\right] + \alpha_h/\lambda_z\,\mathrm{ch}\left[\sqrt{K_a p^2 + s/a_z}\,h\right]}\right\}. \tag{2.2}$$

We now prescribe boundary conditions (2.1) on the surface ($z = 0$) as

$$\left.\frac{\partial \bar{T}(r, z, s)}{\partial z}\right|_{z=0} = \begin{cases} \dfrac{\bar{q}(r, s)}{\lambda_z} & \text{at} \quad r_1 > r > r_0; \\ 0 & \text{at} \quad r_1 < r \leq R \ \text{and} \ 0 \leq r < r_0, \end{cases} \tag{2.3}$$

the heat-flux density $q(r, \tau) = q(\tau)$ (τ is time):

$$\bar{q}_H(p, s) = \frac{\bar{q}(s)}{p}\left[r_1 J_1(pr_1) - r_0 J_1(pr_0)\right], \tag{2.4}$$

$$L\left[q(\tau)\right] = \frac{L\left[Q(\tau)\right]}{\pi(r_1^2 - r_0^2)} = \frac{\bar{Q}(s)}{\pi(r_1^2 - r_0^2)} = \bar{q}(s), \tag{2.5}$$

where $Q(\tau)$ is the absorbed thermal energy from an annular heat source. If $r_0 \to 0$, then formulas (2.4) and (2.5) can be represented in the form

$$\bar{q}_H(p, s) = \bar{q}(s) \frac{r_1}{p} J_1(pr_1) = \frac{\bar{Q}(s)}{\pi r_1 p} J_1(pr_1),$$

where

$$\bar{q}(s) = \frac{\bar{Q}(s)}{\pi r_1^2}. \tag{2.6}$$

Assume that the initial temperature distribution inside the body is constant, $T_0 = \text{const}$. In (2.6), $\bar{Q}(s) = L[Q(\tau)]$ is the absorbed thermal energy from an annular heater in the Laplace transform.

Using Hankel transform (1) for the δ-function in the case of $r_0 < R_c < r_1 < R$, we obtain a value for $\bar{q}_H(p, s)$

$$\bar{q}_H(p, s) = \int_0^R rJ_0(pr)\,\bar{q}(r, s)\,\delta(r - R_c)\,dr = \int_{r_0}^{r_1} rJ_0(pr)\,\bar{q}(r, s)\,\delta(r - R_c)\,dr =$$

$$= R_c J_0(pR_c)\,\bar{q}(R_c, s) = \frac{\bar{Q}(s)}{2\pi} J_0(pR_c), \tag{2.7}$$

where

$$\bar{q}(R_c, s) = \frac{\bar{Q}(s)}{2\pi R_c}. \tag{2.8}$$

In (2.7) and (2.8) heat is transferred across a circumference with radius $r = R_c < R$ on the surface $z = 0$ of a finite orthotropic cylinder with a linear heat-flux density (2.8). The remaining surface $z = 0$, $r < R$ and $r \notin [R_c, R_c + \varepsilon]$ is heat insulated. In (2.7) and (2.8), $\bar{Q}(s) = L[Q(\tau)]$ is the Laplace transform for the absorbed energy of a linear source when its length is $2R_c\pi$.

At $R_c \to 0$ from (2.10) we obtain a value of $\bar{q}_H(p, s)$ on the boundary surface $z = 0$, $r < R$ and $r \notin [0, 0 + \varepsilon]$ for a point heat source at the coordinate origin ($r = z = 0$):

$$\bar{q}_H(p, s) = \frac{\bar{Q}(s)}{2\pi}, \tag{2.9}$$

where $Q(\tau) = L^{-1}[\bar{Q}(s)]$ is the thermal energy of the point heat source.

We obtain corresponding solutions of (2.2) for the interesting cases $\bar{q}_H(p, s)$ by using formulas (2.4), (2.5), (2.7), and (2.9).

Applying solution (2.2) for the case of α_h, $\alpha_k \to \infty$, we arrive at

$$\bar{\theta}_H(p, z, s) = \bar{T}_H(p, z, s) - \frac{T_0 R J_1(pR)}{ps} = \frac{\int_0^R rJ_0(pr)\,\bar{q}(r, s)\,dr}{b_z\sqrt{s + a_r p^2}} \, \frac{\operatorname{sh}\left[\frac{(h - z)}{\sqrt{a_z}}\sqrt{s + a_r p^2}\right]}{\operatorname{ch}\left[\frac{h}{\sqrt{a_z}}\sqrt{s + a_r p^2}\right]}. \tag{2.10}$$

The inverse Hankel transform for solution (2.10) can be obtained by formula (1.8a).

A solution for the inverse transform (2.10) $L[\theta(r, z, s)] = \theta(r, z, \tau)$ is obtained in the following form

$$\theta_H(r, z, \tau) = T_H(p, z, \tau) - T_0 = \frac{a_z}{h\lambda_z} \sum_{n=1}^{\infty} \frac{J_0\left(\mu_n \frac{r}{R}\right)}{J_1^2(\mu_n)} \int_0^{\tau} \exp\left(-\mu_n^2 \frac{a_r\xi}{R^2}\right) \times$$

$$\times\, \theta_1\left(\frac{h - z}{2h}\middle|\, i\pi \frac{a_z\xi}{h^2}\right) \frac{2}{R^2} \int_0^R xJ_0\left(\mu_n \frac{x}{R}\right) q(x, \tau - \xi)\,dx\,d\xi, \tag{2.11}$$

where $\theta_1((h-z)/h^2 | i\pi a_z\xi/h^2)$ is the theta-function [26].

If boundary condition (2.1) on the surface $z = 0$ $(R < r \le 0)$ of a finite orthotropic cylinder is prescribed in the form of (2.3), then solution (2.11) can be represented as

$$T(r, z, \tau) - T_0 = \frac{a_z}{h\lambda_z} \sum_{n=1}^{\infty} \frac{J_0\left(\mu_n \frac{r}{R}\right)}{J_1^2(\mu_n)} \int_0^{\tau} \exp\left(-\mu_n^2 \frac{a_r\xi}{R^2}\right) \times$$

$$\times \theta_1\left(\frac{h-z}{h} \middle| i\pi \frac{a_z\xi}{h^2}\right) \frac{2}{R^2} \int_{r_0}^{r_1} x J_0\left(\mu_n \frac{x}{R}\right) q(x, \tau - \xi)\, dx d\xi, \tag{2.12}$$

where μ_n are the roots of the equation $J_0(\mu_n) = 0$.

At $R \to \infty$, we obtain from (2.12) a solution for an infinite orthotropic plate under initial conditions (2.3) and $T(r, h, \tau) = T_0$:

$$\lim_{R\to\infty} [T(r, z, \tau) - T_0] = T_1(r, z, \tau) - T_0 = \frac{a_z}{2h\lambda_z a_r} \int_0^{\tau} \frac{1}{\xi} \exp\left(-\frac{r^2}{4a_r\xi}\right) \times$$

$$\times \theta_1\left(\frac{h-z}{2h} \middle| i\pi \frac{a_z\xi}{h^2}\right) \int_{r_0}^{r_1} x \exp\left(-\frac{x^2}{4a_r\xi}\right) I_0\left(\frac{rx}{2a_r\xi}\right) q(x, \tau - \xi)\, dx d\xi. \tag{2.13}$$

Solution (2.13) for the Laplace transform $\overline{T}_1(r, z, s)$ is written in the form

$$\overline{T}_1(r, z, s) - \frac{T_0}{s} = \frac{1}{b_z} \int_0^{\infty} \frac{pJ_0(pr)}{\sqrt{s + a_r p^2}} \frac{\operatorname{sh}\left[\frac{(h-z)}{\sqrt{a_z}} \sqrt{s + a_r p^2}\right]}{\operatorname{ch}\left[\frac{h}{\sqrt{a_z}} \sqrt{s + ap^2}\right]} \int_{r_0}^{r_1} xJ_0(px)\, \overline{q}(x, s)\, dx dp. \tag{2.14}$$

At $h \to \infty$ from (2.14) we have the following two-dimensional nonstationary solution for a semiinfinite orthotropic cylinder under initial (1.2) and boundary conditions (2.3):

$$\lim_{h\to\infty} [\overline{T}_1(r, z, s) - T_0/s] = \overline{T}_2(r, z, s) - T_0/s =$$

$$= \frac{1}{b_z} \int_0^{\infty} \frac{pJ_0(pr)}{\sqrt{s + a_r p^2}} \exp\left[\frac{-z}{\sqrt{a_z}} \sqrt{s + a_r p^2}\right] \int_{r_0}^{r_1} xJ_0(px)\, \overline{q}(x, s)\, dx dp. \tag{2.15}$$

Applying the inverse Laplace transformation to (2.15), we obtain the two-dimensional temperature field $T_2(r, z, \tau)$ for a semi-infinite orthotropic cylinder under initial (1.2) and boundary (2.3) conditions:

$$T_2(r, z, \tau) - T_0 = \frac{1}{2\sqrt{\pi}\, b_z a_r} \int_0^{\tau} \frac{1}{\xi^{3/2}} \exp\left[-\left(\frac{r^2}{a_r} + \frac{z^2}{a_z}\right) \frac{1}{4\xi}\right] \times$$

$$\times \int_{r_0}^{r_1} x \exp\left(-\frac{x^2}{4a_r\xi}\right) I_0\left(\frac{rx}{2a_r\xi}\right) q(x, \tau - \xi)\, dx d\xi. \tag{2.16}$$

Solutions and investigations of two-dimensional unsteady heat conduction Problem 2 at $q(r, \tau) = q(\tau)$ over the surface $z = 0$ of different bodies have been carried out in [1-5, 7-18, 20-22, 24, 25].

In [19], an investigation is made of heat transfer in a semiinfinite (in a thermal sense) body when the specific heat flux $q(r, \tau)$ over the body surface $(z = 0)$ in the circular region $(r = r_0)$ is equivalent to $q(r, \tau) =$

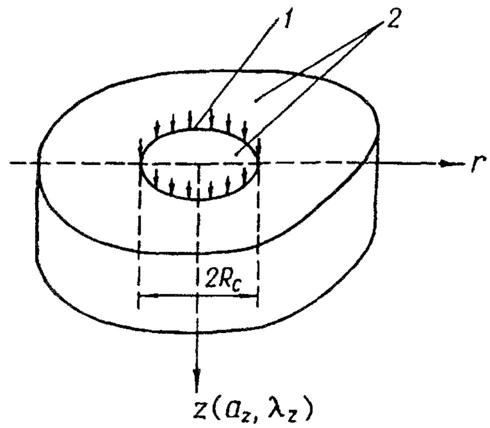

Fig. 3. Idealized physical model of semi-infinite orthotropic body having a source with a heat flux of linear concentration at the periphery ($r = R_c$, $z = 0$): 1) $q_H(p, s) = \overline{Q}(s)/2\pi J_0(p, R_c)$, $r = R_c$; 2) $\partial T(r, 0, \tau)/\partial z = 0$, $r \notin [R_c, R_c + \varepsilon]$.

$\rho W_0(\tau) \exp(-r^2/r_0^2)$, where ρ is the absorptivity, $W_0(\tau)$ is the power of the laser source and r_0 is the radius of a laser beam.

In the present work, Problem 2 (solutions (2.15) and (2.16)) is considered for the following cases.

I. The heat source $q(x, \tau) = L^{-1}[\overline{q}(x, s)]$ is located at the point ($r = z = 0$). Then according to (2.9):

$$\overline{q}_H(p, s) = \frac{\overline{Q}(s)}{2\pi},$$

where $\overline{Q}(s) = L[Q(\tau)] = \int\limits_0^\infty \exp(-s\tau)Q(\tau)d\tau$. Solution (2.15) is written in the form

$$\overline{\theta}_2(r, z, s) = \overline{T}_2(r, z, s) - \frac{T_0}{s} = \frac{\overline{Q}(s)}{2\pi\lambda_z \sqrt{K_a}} \times \times \left\{ \frac{\exp\left(-\sqrt{z^2 K_a + r^2}\sqrt{s}/\sqrt{a_r}\right)}{\sqrt{z^2 K_a + r^2}} \right\}.$$

Determining the temperatures on the surface $z = 0$ at $r = R_1$, $r = R_2$ and their differences $\overline{\theta}_1(s) - \overline{\theta}_2(s) = \overline{T}_2(R_1, 0, s) - \overline{T}_2(R_2, 0, s)$, it is easy to derive a formula for thermal diffusivity a_r:

$$a_r = \frac{(R_2 - R_1)^2}{4\pi} \frac{F_1^2(\tau)}{F_2^2(\tau)},$$

where $(R_2 - R_1)$ is the distance between temperature-sensitive measuring elements on the boundary surface ($z = 0$) of a semi-infinite orthotropic body; $F_1(\tau)$ and $F_2(\tau)$ are determined by the following integrals:

$$F_2(\tau) = \int\limits_0^\tau (\tau - 2\xi)\,\theta_1(\xi)\,\theta_2(\tau - \xi)\,d\xi,$$

$$F_1(\tau) = \int\limits_0^\tau \frac{1}{\sqrt{\tau - t}} \int\limits_0^t \theta_1(\xi)\,\theta_2(t - \xi)\,d\xi dt.$$

Under steady-state conditions ($\tau \to \infty$) it is easy to derive a formula to calculate the complex $\lambda_z\sqrt{K_a} = \lambda_z\sqrt{K_\lambda}$:

$$\lambda_z \sqrt{K_a} = \frac{Q_0}{2\pi\Delta T_2(\infty)} \left(\frac{(R_2 - R_1)}{R_1 R_2}\right),$$

where $\Delta T_2(\infty) = T_2(R_1, 0, \infty) - T_2(R_2, 0, \infty) = \theta_1(\infty) - \theta_2(\infty)$ is the stationary value ($\tau \to \infty$) of the temperature difference at points $r = R_1$, $z = 0$ and $r = R_2$, $z = 0$; Q_0 = const is the constant energy source at the point $r = z = 0$. For an isotropic body $K_a = K_\lambda = 1$, $a_r = a_z = a$, $\lambda_r = \lambda_z = \lambda$.

II. The heat source $q(x, \tau) = L^{-1}[q(x, s)]$ is on the circumference ($r = R_c$, $z = 0$) (Fig. 3).

For this case, solution (2.15) is written as

$$\overline{T}_2(r, z, s) - T_0/s = \frac{\overline{Q}(s)}{2\pi b_z}\int_0^\infty pJ_0(pr)\,J_0(pR_c)\exp\left[-\frac{z}{\sqrt{a_z}}\sqrt{s + a_r p^2}\right]\Big/\sqrt{s + a_r p^2}\,dp.$$

The represented physicomathematical models of unsteady heat conduction with discontinuous boundary conditions on the surface of the tested object allow the development of numerous methods of nondestructive testing of thermophysical properties of various materials.

Local heat fluxes can be formed with the aid both of conventional electric heaters and lasers and electron-beam energy sources. The main difficulty that is encountered in implementation of theoreticophysical measurement by the described methods is that of ensuring precise measurement of surface temperatures at some point or in a definite region on the surface. Recent advances in the design of heat imagers make it possible to measure sufficiently precisely the temperature over the surface by noncontact methods.

The authors believe that the methods of nondestructive testing of thermophysical properties based on the above or similar solutions of the heat conduction equation with discontinuous boundary conditions are most promising and can find application in the design of thermophysical equipment.

Modern microprocessor units equipped with analog-digital converters allow one to automate easily all measurements and calculations.

NOTATION

s, parameter of the integral Laplace transformation; p, parameter of the integral Hankel transformation; a_r, a_z, λ_r, λ_z, K_a, thermal diffusivity and thermal conductivity along cylindrical coordinates; α_h, α_R, heat transfer coefficients on the end and lateral cylinder faces; b_r, b_z, thermal activity; Bi_h, Bi_R, Biot number; $J_0(pr)$, $J_1(pr)$, Bessel functions of the zeroth and first order; $I_0(x)$, modified first-kind zeroth-order Bessel function; $w_0(\tau)$, intensity (density) of laser radiation (W/m^3); ρ, absorptivity of an opaque body; $K_a = a_r/a_z$; $K_\lambda = \lambda_r/\lambda_z$, parameter characterizing the ratios of thermal diffusivities and thermal conductivities in the corresponding directions relative to the cylindrical coordinates r, z; R_1, R_2, inner and outer radii of the circular heat source; $q(r, \tau)$, specific heat flux at $z = 0$; $\theta_1(r, z, \tau) = T_1(r, z, \tau) - T_0$, $\theta_2(r, z, \tau) = T_2(r, z, \tau) - T_0$, excess temperatures; erf cx, probability integral.

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